On the linear and non-linear evolution of dust density perturbations with MOND

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Chapter 1

Introduction

1.1 The standard cosmological model

The standard cosmological model is based on the following assumptions:

- General relativity is valid.
- The cosmological principle is valid.
- The energy content of the universe includes baryonic matter, some form of collisionless matter (dark matter), dark energy with an equation of state given by $\omega = -1$ (cosmological constant) and radiation.

The cosmological principle states that on large scales the universe is spatially homogeneous and isotropic (see, for instance, Wald (1984) for a precise definition of this terms). Under this assumption, the solution of the Einstein's equation can be written as:

$$g_{\mu\nu} = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{1.1}$$

which is known as the Friedman-Robertson-Walker (FRW) metric. The quantity a is called *expansion factor* and k is a constant, measure of the curvature of the universe.

One of the basic tests that was made of the model consist of measuring the ratio $H(a) = \dot{a}/a$ by means of supernovae observations (e.g. Riess et al. 1998). The data is consistent with the FRW metric and with the fact that the constant k is equal to 0 (i.e. the universe is flat).

Another classical test is based on high redshift observations. The fact that the FRW metric predicts the existence of a singularity at $t = 0$ induces the idea of a hot big bang. At very early times, owing to the very high temperature, matter and radiation were coupled via electron scattering and followed the same distributions. The temperature decreases with the adiabatic expansion of the universe; when the temperature falls below the value needed for the plasma to recombine, the radiation decouples from the matter and becomes capable of freely traveling. Once the radiation is released, it is redshifted because of the expansion. In consequence, this primordial distribution of photons is possible to observe today as black body radiation corresponding to a temperature of 3K. This strong prediction was successfully confirmed in two complementary works in 1965 (Penzias & Wilson 1965; Dicke et al. 1965) when the cosmic microwave background (CMB) was observed for the first time.

On small scales, the cosmological principle is not valid and the matter density presents irregularities that are assumed to have originated as quantum fluctuations during an epoch of exponential growth called inflation. These initial inhomogeneities can be observed at the CMB and can be employed to estimate the cosmological parameters. One possible set of parameters that was obtained using a combination of CMB observations and supernovae data is: $\Omega_b = 0.0449$, $\Omega_{\rm cdm} = 0.222$, $\Omega_\Lambda = .734$ (Larson et al. 2010), that correspond to the density of the baryonic matter, cold dark matter and dark energy components normalized with the critical density of the universe.

The existence of inhomogeneities allows matter to collapse under the effects of gravity, eventually forming galaxies, planets, human beings, etc. The topic of this thesis is the time evolution of such inhomogeneities under a special family of gravitational theories. The solutions for the standard case will be taken as reference and used in part for the calculations and thus, they will be described in detail in the following section.

Dark matter and dark energy

The distribution of energy presented in this last paragraph, indicates that most of the energy content of the universe is in its unseen components (dark matter and dark energy). A few words must be given about them.

The concept of dark matter was introduced for first time in the context of dynamics of clusters of galaxies. The first attempts to explain the dynamics of galaxies using Newtonian gravity were made by Zwicky (1933)[1]. After making a kinematic analysis of the cluster of galaxies Coma, Zwicky found a large discrepancy between the observed values of the velocity dispersion and those that were predicted by the theory. He arrived to a conclusion that can be stated in two completely different ways according to the preconceptions made:

- Newtonian gravity has serious problems to explain clusters.
- Newtonian gravity predicts the existence of a large amount of unseen matter.

(Zwicky only mentioned the dark matter option).

The other dark component, dark energy, was introduced in the decade of 1990 (e.g. Riess et al. 1998), when supernovae observations employed to measure the rate of expansion of the universe showed that, at the present epoch, the expansion is being accelerated. The effect can be mathematically stated as a modification of the Einstein's equation, which becomes:

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} + g_{\alpha\beta}\Lambda, \tag{1.2}$$

where the *cosmological constant* Λ takes into account the presence of this new component. While the new term helps to cure the problems with the expansion, there are still two open questions related to it, which could be the signal that the gravitational theory has to be modified. In first place, the value of the cosmological constant can be estimated in

[1]English translation can be found in Zwicky (2009)

the context of quantum field theory. The predicted number is about 10^{120} times greater that the observed one. The other problem is related to the fact that $\Omega_{\rm b}+\Omega_{\rm cdm}+\Omega_\Lambda\sim 1$. The quantity $\Omega_{\rm b}+\Omega_{\rm cdm}$ evolves with time as $(1+z)^3$ (see section 1.2.1), while Ω_Λ is a constant by definition. The question that immediately arises is the following one: why are we observing at such a special epoch in which two quantities with such different time evolution have the same order of magnitude?

Friedmann's equations from Newton's formula

The set of equations that describes the behavior of the function a in eq.1.1, the so called Friedmann equations, can be obtained using a set of arguments based almost purely in Newtonian physics. As this kind of reasoning will be employed in the following sections to obtain equations for MOND, it will be briefly discussed here for the standard case.

The Newtonian method used to get expressions for the rate of expansion of spatially homogeneous and isotropic universes is the following one: take a spherical region of the infinite isotropic and homogeneous universe and assume that the Birkoff theorem is valid. In that case, the acceleration at the boundary of the sphere is determined by the enclosed mass and given by:

$$\ddot{r}=-\frac{GM}{r^2}, \tag{1.3}$$

where

$$M=\frac{4}{3}\pi r^3(\rho+3P), \tag{1.4}$$

where $\rho+3P$ is the active gravitational mass. A cosmological constant term responsible for the acceleration of the expansion can be introduced as the presence of a fluid with negative pressure:

$$P_\Lambda=-\frac{\Lambda/3}{4\pi G}. \tag{1.5}$$

Substitution of the mass into the equation of motion gives:

$$\ddot{r}=-\frac{4\pi G}{3}(\rho+3P_T)r+\frac{\Lambda}{3}r, \tag{1.6}$$

where P_T is the thermodynamic pressure of the fluid.

The spatial variable can be rewritten as a function of a fixed length scale:

$$r(t)=r_0a(t), \tag{1.7}$$

where r_0 is the position of the particle at some initial time and $a(t)$ is a function independent of r_0. Substituting this definition into the equation of motion (eq.1.6) gives an equation for $a(t)$:

$$\ddot{a}=-\frac{4\pi G}{3}(\rho+3P_T)a+\frac{\Lambda}{3}a, \tag{1.8}$$

which is the second order Friedmann equation. A first order equation can be obtained by substituting the pressure that appears in the last expresion by the value provided by the continuity equation:

$$\dot{\rho}=-3(\rho+P_T)\frac{\dot{a}}{a}. \tag{1.9}$$

This gives an equation whose integrating factor is $\dot{a}$. After integrating once, one obtains:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} + K, \tag{1.10}$$

which is the second Friedmann equation. While the derivation was not made from basic principles, the equations are correct and provide a zero order solution for the evolution of the matter density through cosmic time. The first order solutions for a perturbed density distribution will be described in the next section, as well as a re-derivation of the Friedmann equations from basic principles.

1.2 Evolution of dust density perturbations through cosmic time

The evolution of density perturbations through cosmic time can be studied in two different regimes, which are known as linear and non-linear. The term linear makes reference to the fact that the analysis, owing to reasons that will be described in brief, is valid only for small perturbations of the density. While this approach seems to be restrictive, it has the advantage of providing equations that have analytic solutions. These approximate solutions can be employed a posteriori as a reference point for understanding the underlying physics behind the non-linear results.

The equations that describe the non-linear evolution must be solved numerically by means of, for instance, N-body algorithms. As similar techniques will be employed during this thesis, the set of equations will be described in detail for the standard case. A few technical properties of one of the possible implementations of the solution will also be discussed. Both formalisms (linear and non-linear) make use of the linearized Einstein's equation; its form and some properties of its solution will be also described.

1.2.1 The linearized Einstein's equation with an expanding background

We propose as solution for the Einstein's equation, a metric constructed as a scalar perturbation of the flat case (k=0) of the FRW metric (eq.1.1):

$$ds^2 = -a(t)^m(1+2\phi)dt^2 + a(t)^n(1-2\phi)(dx^2+dy^2+dz^2). \tag{1.11}$$

The two free parameters (m, n) give some freedom about the gauge in which the metric is written. Different sets of coordinates are usually used in the context of cosmological evolution, therefore, it is convenient at this point to study this general solution. The cases of interest will be $(m, n) = (0, 2)$, which corresponds to the Newtonian gauge (comoving coordinates in the language of cosmological simulations) and $(m, n) = (4, 2)$, associated to supercomoving coordinates (Martel & Shapiro 1998).

The perturbed energy momentum tensor is given by:

$$T_{ab} = (\rho_0 + \delta\rho)u_a u_b + (P_0 + \delta P)(g_{ab} + u_a u_b) \tag{1.12}$$

where ρ_0 is the background density and $\delta\rho$ is the perturbation on the density that is responsible for the perturbation assumed on the metric ϕ. The four-velocity vector can be written as:

$$u^a = [A, v^1, v^2, v^3], \tag{1.13}$$

where v^i are the spatial velocities of the particles and A is determined by requiring:

$$g_{ab}u^a u^b = -1. \tag{1.14}$$

Neglecting quadratic terms in the velocities, the four-velocity vector becomes:

$$u^a = [1 - \phi, v^1, v^2, v^3]. \tag{1.15}$$

and substituting the quantities $g_{\mu\nu}$ and $T_{\mu\nu}$ into the Einstein's equation gives, in the quasi-static limit ($\partial\phi/\partial t \ll 1$), the following zero and first order equations:

$$H^2 = \frac{4a^m}{n^2}\frac{8\pi G}{3}\rho_0 \tag{1.16}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{n}a^m P_0 + \frac{2m - 3n + 4}{4}H^2 \tag{1.17}$$

$$\nabla^2\phi = 4\pi G a^n \delta\rho, \tag{1.18}$$

which are the Friedman equations described before and the equivalent to the Poisson equation in the expanding context. As the calculation was made in the quasi-static limit, the resulting Poisson equation is independent of the constant m.

The equation for the time evolution of the background density can be derived in a straightforward way from the Friedmann equations. The resulting expression is the following one:

$$\dot{\rho_0} + \frac{3n}{2}H(P_0 + \rho_0) = 0. \tag{1.19}$$

which is also independent of m; therefore, the background has the same time evolution in both coordinate systems. In the case of dust, $P = 0$ and the solution is given by:

$$\rho_0 \propto a^{-3n/2}. \tag{1.20}$$

For radiation, the pressure behaves as $P_0 = \rho_0/3$ and the solution becomes:

$$\rho_0 \propto a^{-2n}. \tag{1.21}$$

The second order Friedmann equation (eq.1.17) is commonly re-written as:

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{n}a^m \left[P_0 - \frac{2m - 3n + 4}{3n}\rho_0\right], \tag{1.22}$$

which is a consequence of substituting the term that depends on H in eq.1.17 from the expression given by the first Friedmann equation (eq.1.16). Note that this form of the equation is the one that was found in previous section from the Newtonian set of arguments.

1.2.2 Linear regime

The time evolution of dust perturbations can be described by the equation of motion for matter (the conservation of energy) coupled to the Einstein equation:

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{1.23}$$
$$\nabla^{\alpha} T_{\alpha\beta} = 0. \tag{1.24}$$

Projecting the eq.1.24 in the direction parallel and perpendicular to u^a gives:

$$u^a \nabla_a \rho + (\rho + P)\nabla^a u_a = 0 \tag{1.25}$$
$$(P + \rho) u^a \nabla_a u_b + (g_{ab} + u_a u_b)\nabla^a P = 0. \tag{1.26}$$

The plan is to take the first order and quasi-static limit over a FRW background of these two equations; they will result in the continuity and Euler's equation. A linear combination of these two equations plus substitution of the linearized Einstein equation will finally give the so called *growth equation* which describes the *linear* time evolution of the density perturbations. Pressure effects will be assumed as higher order and the coordinates will be such that $(m, n) = (0, 2)$.

Continuity equation

By substituting the four velocity u^a given by eq.1.15 and the perturbed density ($\rho = \rho_0 + \delta\rho$) in eq.1.25, making the derivatives using the metric given by eq.1.11 and taking into account only terms up to first order, one gets:

$$\left[\frac{\partial \rho_0}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} - 3H\rho_0\right] - \left[\frac{\partial \rho_0}{\partial t} - 3H\rho_0\right]\phi + \left[\frac{\partial \delta\rho}{\partial t} + \nabla \cdot (\delta\rho \mathbf{v}) + 3H\delta\rho\right] = 0. \tag{1.27}$$

The dot product, the nabla operator and the velocity $\mathbf{v}$ should be interpreted only in 3D space. To assume that the perturbation is zero, shows that the first square bracket is equal to zero. The second squared bracket is also zero (eq.1.19), so the continuity equation for the perturbed quantities is:

$$\frac{\partial \delta\rho}{\partial t} + \nabla \cdot (\delta\rho \mathbf{v}) + 3H\delta\rho = 0. \tag{1.28}$$

The last expression can be written in a different form as a function of the *peculiar velocity* $\mathbf{u}$ which is defined by:

$$\mathbf{u} = a\mathbf{v} \tag{1.29}$$

and which is the term unrelated to the expansion when the velocity is written according to the variables used in the Newtonian approach:

$$\dot{\mathbf{r}} = a\dot{\mathbf{v}} + H\mathbf{r}. \tag{1.30}$$

In terms of this velocity, the continuity equation reads:

$$\frac{\partial \delta\rho}{\partial t} + \frac{1}{a}\nabla \cdot (\delta\rho \mathbf{u}) + 3H\delta\rho = 0. \tag{1.31}$$

A very useful relation between the *overdensity*

$$\delta = \delta\rho/\rho_0 \tag{1.32}$$

and the velocity field can be obtained by writting the density as:

$$\rho = \rho_0(1+\delta), \tag{1.33}$$

and taking the first order in eq.1.27. The resulting relation reads as follows:

$$\frac{\partial \delta}{\partial t} + \nabla \cdot \mathbf{v} = 0. \tag{1.34}$$

Euler's equation

The cosmological Euler's equation

$$\frac{\partial v_i}{\partial t} + v^j \frac{\partial v_i}{\partial x^j} + H v_i = \frac{\partial \phi}{\partial x^i} \tag{1.35}$$

can be obtained by applying a procedure similar to the one used in previous paragraph to the equation 1.26.

In terms of the velocity defined by eq.1.29, the Euler's equation becomes:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{a}(\mathbf{u}.\nabla)\mathbf{u} + 2H\mathbf{u} = \frac{1}{a}\nabla\phi. \tag{1.36}$$

Growth equation

Finally, the growth equation for density perturbations:

$$\ddot{\delta} + 2H\dot{\delta} - \frac{\nabla^2\phi}{a^2} = 0 \tag{1.37}$$

can be found by adding the divergence of eq.1.35 to eq.1.28 and substituting the definition of the overdensity (eq.1.32).

After taking into account the Poisson equation (eq.1.18), one finally gets:

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_0\delta = 0. \tag{1.38}$$

An interesting feature of the growth equation is that there are no derivatives with respect to the position and that it is homogeneous. This will give separable solutions of the form:

$$\delta(\mathbf{x}, t) = D(t)\delta_0(\mathbf{x}), \tag{1.39}$$

where $D(t)$ is the so called *growth factor* and $\delta_0(\mathbf{x})$ is the initial density given at some initial time t_0. The evolution is self-similar, presenting changes of the amplitude of the perturbation but not of its spatial dependence.

Analytic solutions for the growth factor can be found in the following way. The growth equation is a second order ordinary differential equation, for which two independent solutions are needed. By substitution into eq.1.38, it is trivial to see that

$$D_1(a) = H(a) \tag{1.40}$$

is a solution. The other solution can be derived by converting the equation into a system of first order differential equations. By taking in account Liouville's formula for the Wronskian and the already known solution one gets a first order differential equation whose integrating factor is H^{-1}. After making a change of variables from t to a, one has that the formal solution obtained with this procedure is:

$$D(a) = \frac{H(a)}{D_0} \int_0^a \frac{1}{\dot{a}^3} da, \tag{1.41}$$

where D_0 is a constant. For an Einstein-de Sitter universe ($\Omega_{\rm m} = 1$, $\Omega_\Lambda = 0$, $\Omega_{\rm r} = 0$) the solution behaves as $D(a) = a$ in the matter dominated era.

The definition of the overdensity δ provides information about the domain of validity of the solution: as the density ρ has to be positive, the overdensity must be bound from below by -1. This restriction is not taken into account by the growth equation. In consequence, its solutions are incorrect for values of t such that $\delta < -1$. Of course, as any analysis based on perturbation theory, the approximations are expected to break down before this value is reached, but this reasoning gives a hard boundary that can not be crossed.

It is important to emphasize that the presence of scale independent solutions is not a synonym for linear evolution. While this is valid for standard gravity, it is not a general result and in particular, it is not valid for MOND. This is a fundamental difference between Newtonian and MONDian gravity that will be discussed in detail throughout this thesis.

The need for dark matter

The present analysis was made for perturbations evolving during the matter dominated era. A similar calculation extended into the radiation dominated era pre-recombination (Silk 1968) shows that, for a universe populated only with baryons and radiation, the perturbations are damped by interaction between the electrons and the photons. It is possible to construct theories of galaxy formation under this conditions (e.g. Zel'Dovich 1970), but the overdensity that is needed on clusters scales to make galactic scales to collapse is of order $\delta \sim 10^{-3}$, which is in contradiction with the observations of temperature fluctuations at the CMB. The problem can be solved by including a dark matter component to balance the Silk damping. This new component, which is assumed to be the same dark matter proposed by Zwicky, will permit the collapse on small scales and at the same time will give the right amplitude on large scales.

1.2.3 Non-linear regime

At the moment in which the density perturbations are of the order of 1, the approximations that were made to study the linear evolution break down and a different approach must be implemented. A standard way to do this, is to use N-body simulations, which consist in describing the density by means of a set of particles whose positions evolve according to the geodesic equation. In other words, instead of describing the evolution of matter density through eqs.1.23 and 1.24, the following equations are employed:

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.42}$$

$$\frac{d^2 x^\sigma}{d\tau^2} + \Gamma^\sigma_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0. \tag{1.43}$$

The solutions of this equations can be found, again, by applying perturbation theory. By substituting the perturbed metric given by eq.1.11 into eq.1.43 and parametrizing the solution with time in the Newtonian gauge one gets for the component x:

$$\ddot{x} + 2\frac{\dot{a}}{a}\dot{x} + \frac{1}{1+2\phi}\frac{\partial\phi}{\partial x}\left(\frac{1}{a^2} + \dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{2a^2}{1+2\phi}\left(\frac{\partial\phi}{\partial y}\dot{x}\dot{y} + \frac{\partial\phi}{\partial z}\dot{x}\dot{z}\right) = 0 \tag{1.44}$$

and similar equations for the other two spatial components. After neglecting second order terms, the equation becomes:

$$\ddot{x} + 2\frac{\dot{a}}{a}\dot{x} + \frac{1}{a^2}\frac{\partial\phi}{\partial x} = 0. \tag{1.45}$$

Furthermore, the following definition:

$$p^i = a^2 \dot{x}^i \tag{1.46}$$

gives a set of first order equations which resembles the Hamilton's equations used in classical dynamics:

$$\begin{aligned} \dot{x}^i &= \frac{p^i}{a^2} \\ \dot{p}^i &= -\frac{\partial\phi}{\partial x^i}. \end{aligned} \tag{1.47}$$

These are the equations that, coupled to the linearized Einstein's equation, will be solved by means of the N-body algorithm.

The last point that must be specified before obtaining a closed set of equations is the form in which the description of the density is made through the particles. To this end, let us take a cubic piece of the universe of side length B of several megaparsecs and calculate the density on a grid of size h. In the case that the number of particles is infinite, the density in the cell (i, j, k) can be defined as:

$$\rho_{i,j,k} = \frac{1}{h^3}\sum_n m_n, \tag{1.48}$$

where m_n is the mass of the particle n and the index n runs over all the particles in the cell. The size of the cells should tend to zero in the hydrodynamical sense. As in practice, the number of particles that is employed in the simulations is far smaller than the one needed to make this approximation to be valid, a smoothing technique must be implemented. The initially point-like masses are now assumed to have a finite size and to contribute not only to the mass of the cell to which they belong but also to the neighboring cells.

Many smoothing techniques are present in the literature (e.g. Hockney & Eastwood 1988). One example is the triangular shape cloud (TSC) that ensures continuity of the density up to its second derivative and for which the estimated density on the node (i, j, k) is given by:

$$\rho_{i,j,k} = \frac{1}{h^3} \sum_n W_1(x^1_{i,j,k} - x^1_n) W_2(x^2_{i,j,k} - x^2_n) W_3(x^3_{i,j,k} - x^3_n) m_n, \tag{1.49}$$

where W is the smoothing kernel which is given by:

$$W(x) = \begin{cases} \frac{3}{4} - \left(\frac{x}{h}\right)^2 & \frac{h}{2} <= \frac{|x|}{2} \\ \frac{1}{2}\left(\frac{3}{2} - \frac{|x|}{h}\right)^2 & \frac{h}{2} <= |x| <= \frac{3h}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{1.50}$$

The index on the summation n runs in this case over all the particles belonging to the system.

To summarize, the methodology used to study non-linear evolution consist of the integration of the equations of motion (eqs.1.47) for a large set of particles coupled to the field equation (eq.1.18). The source of this last equation is calculated from the position of the particles using eq.1.49. The number of particles is limited by the computational resources and goes from 32^3 particles in the first simulations that appeared in the literature to 1024^3 for the state of the art simulations made with standard gravity. The integration of the equations of motion can be done using a leap-frog scheme (e.g. Hockney & Eastwood 1988) given by:

$$\begin{aligned} v_{n+1/2} &= v_{n-1/2} - \frac{(\nabla\phi)_n}{\dot{a}_n}\Delta a \\ x_{n+1} &= x_n + \frac{v_{n+1/2}}{\dot{a}_{n+1/2} a^2_{n+1/2}}\Delta a, \end{aligned} \tag{1.51}$$

where Δa is the size of the time steps and the expansion factor a is used as the time variable. Since the Poisson equation is linear, the calculation of the potential on the grid can be made using FFT methods (e.g. Hockney & Eastwood 1988). For the modified case, the non-linearity of the MOND equation will enforce the use of different methods, which will be described in detail in chapter 2. Once the gravitational potential has been computed, the forces are calculated in first place on the grid using discretization formulas for the derivatives of the potential and then interpolated to the position of the particles. To ensure momentum conservation, this interpolation must be made by means of an interpolation scheme based on the same kernel used to calculate the density (e.g. Gross 1997).

One could ask at this point why we are talking about non-linear evolution at the same time that we are proposing to study linearized equations. The answer resides in which is the quantity that is assumed to be small when making the linearization. In the linear regime (section 1.2.2) the over-density δ is assumed to be small, while here, the constrain is put on the spatial and time derivatives of the metric and the velocity. In other words, in the non-linear evolution, the system is assumed to be non relativistic in the sense of special relativity, no matter what the density is. The approximation will fail close to black holes, but it will still be valid inside galaxies, where the over-densities are greater than one and hence the approximations made in last section break down.

Classical results from N-body cosmological simulations are related to the evolution of the power spectrum and the correlation function in the non-linear regime and its consequences on the initial power spectrum (Efstathiou & Eastwood 1981; Hamilton et al. 1991). It has been proved for instance that the amplitude of the correlation function has a self-similar behavior also in the non-linear regime after the formed objects have been virialized and are fully decoupled from the expansion. The increment in the size of the computers has permitted to increase the number of particles in the simulations and made possible the study of the properties of the collapsed objects. Navarro et al. (1996, 1997) have found that the density distribution of the virialized dark matter haloes follows a universal profile which presents a central singularity with inner and outer logarithmic slopes of -1 and -3 respectively. Regarding the substructure present in the dark matter haloes corresponding to the size of the Milky Way halo, Klypin et al. (1999) have found, also using N-body simulations, an excess in the number of substructure with respect to the number of observed satellites galaxies. Many solutions have been proposed, all of them invoking some astrophysical mechanism to shut down stellar formation in small halos preventing the formation of dwarf galaxies and at the same time conserving the predicted number of subhalos (e.g. Hoeft et al. 2006).

Later on, baryonic physics was added to the calculations as well as the effect of many astrophysical processes: cooling, star formation, supernovae feedback and chemical enrichment, all of them given by a variety of recipes motivated by different models. At present, the main interest has moved from the properties of the large structure to the formation of galaxies. For instance, great effort is invested regarding the formation of the disk of the Milky Way, where there are still open problems. One of them is related to the angular momentum of the stellar component (e.g. Steinmetz 1999): a common feature of the simulations is that there is a transport of angular momentum from the baryons to the dark matter halo, preventing the existence of large disks as observed in the Milky Way. Other problems are related to the characteristics of the galaxies that are obtained in these simulations, as for instance the ratio between the masses of the bulge and the disc (e.g. Scannapieco et al. 2009).

1.3 MOdified Newtonian Dynamics

Modifications of the gravitational theory are not new. Our opinions about how to describe and to explain gravitational phenomena has a long history of evolution and modifications on modifications. Many of these modifications have survived until now and many others were found to be non viable just after being presented. The first gravitational theory was proposed by Newton as a fitting formula for solar system observations (Newton 1686)[2]. It is given in its modern form by the Poisson equation:

$$\nabla^2 \phi_N = 4\pi G\rho, \tag{1.52}$$

where ρ is the matter density and ϕ_N is the gravitational potential from which the gravitational forces are defined as

$$\mathbf{g}_N = \nabla \phi_N. \tag{1.53}$$

[2]English translation of this work can be found in Newton (1934).

The *gravitational constant* G was assumed to be independent of time and space and its value had to be measured in laboratory, with no further motivation from basic principles. Test particles under the influence of gravity move according to Hamilton's equations:

$$\dot{\mathbf{x}} = \mathbf{p} \tag{1.54}$$
$$\dot{\mathbf{p}} = \nabla\phi_N. \tag{1.55}$$

The first modification accepted by the community was proposed by Einstein (1915) and substitutes the concept of gravitational force by the concept of curvature of space-time. The field equation for the metric $g_{\mu\nu}$ that describes the curvature can be obtained with a variational principle applied to the following Lagrangian:

$$L = R\sqrt{-g}, \tag{1.56}$$

where R is the contracted Ricci tensor and g is the determinant of the metric. The resulting field equations is:

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{1.57}$$

where c is the speed of light which, in the same way as the gravitational constant G, has to be measured in laboratory. As gravity is not a force anymore, test particles under its the influence are considered as free and move following the geodesics equation:

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0. \tag{1.58}$$

Details about those equations will not be discussed here and can be found in any basic book on general relativity (e.g. Wald 1984).

The first modifications to this law started to appear a couple of years after Einstein's proposal (Weyl, 1919; Eddington 1922); the main motivation at that moment was to give a geometrical explanation to electromagnetism. Einstein himself also made changes, proposing to add a cosmological constant to his theory in order to obtain static solutions for the universe. Later on, when the expansion of the universe was discovered (Hubble 1929), Einstein rejected his own idea; but new observations in the decade of 1990 (e.g. Riess et al. 1998) showed that the expansion is being accelerated and revived the cosmological term to be accepted as part of the standard gravitational theory.

Other modifications were proposed in the mean time. Examples of such theories are those that can be obtained giving freedom on the dependence with the Ricci scalar to the Lagrangian:

$$L = f(R)\sqrt{-g}. \tag{1.59}$$

This family of theories was born as a possible way to avoid the initial singularity predicted by general relativity for the universe (Buchdahl 1970) and has a great impetus in recent times as a way to explain the peculiar behavior of accelerated expansion. For a review on this kind of theories see for instance Sotiriou & Faraoni (2008). Other modifications where proposed by adding complementary fields besides the metric, giving rise to vector-tensor theories (e.g. Will & Nordtvedt 1972; Hellings & Nordtvedt 1973), scalar-tensor theories (e.g. Bergmann 1968; Wagoner 1970; Brans & Dicke 1961) or bi-metric theories (Rosen 1973; Milgrom 2009a). Motivations for this modifications are multiple: unification between gravity and electromagnetism, quantization of gravity, fundamental explanation of the cosmological constant and also pure theoretical exercises.

Given this panorama of theories, one can see that all of them have something in common: they are motivated as modifications of general relativity, which can be interpreted as an extension of Newtonian gravity in the high acceleration regime. Regarding this point, one of the classical books on modifications of gravity (Will 1993) states in an explicit way that, in order to be considered as viable, any gravitational theory needs to approach the Newtonian theory when the accelerations are low. Nevertheless, one can always ask about the validity of this assertion. What if Newton was wrong? Is it possible that theoreticians are trying to improve a theory (general relativity) which is already wrong in its non-relativistic limit? One should always keep in mind that Newton did not know about galaxies nor about cosmology. His theory was developed to explain the solar system; any extrapolation into galactic dynamics or even cosmology should be made with extreme care. Newton's explanations of galaxies can not be taken as more than predictions that have to be tested in an independent way before taken as valid. Milgrom's theory was proposed under this line of thinking. He decided to make the change not in the relativistic framework as the community was doing at that moment, but to go one step backwards and make the change in the Newtonian framework.

The Modified Newtonian Dynamics (MOND) was presented in Milgrom (1983) as an alternative to the dark matter paradigm. The idea was to assume the non existence of the dark component in the universe and to make a modification in the law of gravity to explain the kinematics of galaxies. The original proposal was to calculate the gravitational forces in the following way[3]:

$$\mu\left(\frac{|\nabla\phi|}{a_0}\right)\nabla\phi = \nabla\phi_N, \tag{1.60}$$

where a_0 is a new fundamental constant with units of acceleration, which, as well as G, has to be measured experimentally. The potential ϕ_N is the Newtonian potential solution of eq.1.52 and ϕ is the *MONDian potential* which is the one that should be used in the equations of motion (eq.1.55) to obtain trajectories of particles. The *interpolating function* μ connects the regime of high and low accelerations and has the following asymptotic behavior:

$$\mu = \left[\begin{array}{ll} x & \text{if } \nabla\phi \ll a_0 \\ 1 & \text{if } \nabla\phi \gg a_0. \end{array}\right. \tag{1.61}$$

The observed value for a_0 from rotations curves of galaxies is on the order of 10^{-8} cm/sec^2 (Begeman et al. 1991), which makes Milgrom's and Newton's theory to be equivalent in the context of the dynamics of the planets in the solar system. It is interesting to note that $a_0 \sim c\sqrt{\Lambda}$, which could be a signal that there is more fundamental physics behind this new fundamental constant. There is another similarity between a_0 and already known parameters: $a_0 \sim cH_0$. In case that this is not just a coincidence, it will imply that a_0 has, like the Hubble constant H, a time dependence. As well as a_0, the μ function is guessed by studying kinematic of galaxies; the form presently used (Zhao & Famaey 2006) in most of the studies is given by:

$$\mu(x) = \frac{x}{1+x}. \tag{1.62}$$

[3]The formula can be interpreted also as a modification the of the law of inertia instead of gravity. This alternative point of view will not be discussed during this thesis.

For a review on the different μ functions that have been considered see McGaugh (2008). It is important to note that to talk about high accelerations in a MONDian context, means to talk about accelerations greater than a_0, but still much smaller than the accelerations needed for relativistic corrections to be important.

There is an essential difference between MONDian and Newtonian potentials that can be found in systems as simples as a point mass. The asymptotic behavior of this solution is given by:

$$\phi(r) = \left[\begin{array}{rl} \sqrt{GMa_0}\ln r & \text{if } \nabla\phi \ll a_0 \\ GM/r & \text{if } \nabla\phi \gg a_0. \end{array} \right. \tag{1.63}$$

As the potential diverges at infinity, there is no escape velocity for isolated systems.

It is easy to show that the original formulation for MOND does not conserve momentum (Felten 1984), which is a desirable property of any theory. The problem was solved by Bekenstein & Milgrom (1984), who derived the theory from a variational principle that gave rise to the following field equation for the MONDian potential:

$$\nabla \cdot \left[\mu \left(\frac{\nabla\phi}{a_0} \right) \nabla\phi \right] = 4\pi G\rho, \tag{1.64}$$

which will be called during this thesis as Bekenstein-Milgrom (BM) equation. An alternative way of writing this equation is the following:

$$\mu \left(\frac{\nabla\phi}{a_0} \right) \nabla\phi = \nabla\phi_N + \nabla \times \mathbf{k}, \tag{1.65}$$

where $\nabla \times \mathbf{k}$ is the *curl field* that converts eq.1.60 in a conservative equation. It is possible to prove (Bekenstein & Milgrom 1984) that the curl field is exactly zero under spherical and cylindrical symmetry and that it behaves as $1/r^3$ for non-symmetric situations. This property gives the opportunity to obtain approximations for the MONDian forces once the Newtonian solution has been determined. An example of such a solution that will be used later in this chapter is the one obtained in the deepMOND regime ($\nabla\phi \ll a_0$):

$$\nabla\phi = \sqrt{GMa_0}\nabla\phi_N. \tag{1.66}$$

Nevertheless, there are non-negligible differences between the dynamics described by the simple version (eq.1.60) and the full non-linear one (eq.1.64). The impact that this term has in non-linear structure formation will be studied in detail in chapter 2.

The MOND theory is extremely successful in explaining the rotation curves of galaxies as well as the Tully-Fisher relation in the absence of CDM (McGaugh et al. 2000; McGaugh 2005). In fact, MOND successfully matches the observations on a wide range of scales, from globular clusters (Gentile et al. 2010) to different types of galaxies including dwarfs and giants, spirals and ellipticals (Milgrom 2007; Gentile et al. 2007; Milgrom & Sanders 2007; Famaey & Binney 2005; Sanders & Noordermeer 2007; Angus 2008).

On scales larger than galaxies, the original formulation of MOND (BM equation plus a universe containing only baryons) gives predictions that disagree with observations. Sanders (2003) found a discrepancy in the context of galaxy clusters and proposed to include a dark matter component to solve it. As this dark component is needed in clusters but not in galaxies, it must be constituted by particles that are unable to collapse on the scale of galaxies; a condition that is fulfilled by neutrinos. It was proposed to use classical neutrinos with a mass of 2eV as the dark matter component (Sanders 2003), but Natarajan & Zhao (2008) gave indications that this kind of neutrino is not enough to explain the observations. In a cosmological context, McGaugh (2004) pointed out a possible discrepancy between observations of the angular power spectrum of perturbations at the CMB and the predictions made for a universe populated only with baryons. Angus (2009) proposed a model with sterile neutrinos which is able to explain both CMB observations (Angus 2009) and cluster of galaxies (Angus et al. 2009), although the existence of this particles remains hypothetical.

Naturally, for Milgrom's idea to become a proper theory, it has to be able to explain relativistic phenomena: gravitational waves, black holes, expansion of the universe, gravitational lensing, etc. The first efforts to extend MOND into the relativistic regime come from the very beginning (Bekenstein & Milgrom 1984), when the RAQUAL theory was proposed. Many theories are present at this moment, all of them working with extra fields besides the metric. There are vector-tensor theories (e.g. Zlosnik et al. 2007; Halle et al. 2008; Blanchet & Le Tiec 2008), vector-scalar-tensor theories (Sanders 1997; Bekenstein 2004) and bimetric theories (Milgrom 2009a). Some of these theories are already known to be non-viable but they represent the basis for more complex theories, as for instance TeVeS, which took some ideas of previous theories. There is still a lot of work to do in this direction. For instance, classical test as preference frame PPN parameters have not yet been calculated for some of them.

1.4 Cosmology with MOND

As MOND was proposed in a non-relativistic context and still there is no unique well established relativistic extension, it is not possible at present to study cosmology in a fully unique self-consistent way. Nevertheless, at least in standard cosmology, it is possible to extract information about the evolution of regions of the universe smaller than the horizon using purely Newtonian arguments. Repeating these arguments in the context of MOND, gives a set of Friedmann's equations that is not well behaved. Some particular relativistic extensions, show solutions with a behavior close to the standard model. From the point of view of non-linear evolution, only very few studies were made, using simplified expressions for MOND.

1.4.1 MONDian expansion

Non-relativistic treatment

The equation of motion for a particle located in an spatially homogeneous and isotropic universe can be written considering that it is situated on the surface of a sphere of radius r and following the analogy of Newtonian cosmology. In the deep MOND regime (see eq.1.66), the equation is the following one:

$$\ddot{r} = \sqrt{\frac{a_0 GM}{r^2}}, \tag{1.67}$$

which taking into account the expression for the mass given by eq. 1.4 becomes:

$$\ddot{r} = \sqrt{\frac{4\pi G}{3} a_0 r \left[\rho + 3(P_T + P_\Lambda)\right]}, \tag{1.68}$$

where P_T is the thermodynamics pressure and P_Λ is the pressure associated to the cosmological constant defined in eq.1.5. The Λ term can be extracted from the square root by means of a Taylor expansion. Making the change given by eq.1.7 and using the definition 1.5 for the dark energy pressure gives:

$$\ddot{a} = \sqrt{\frac{4\pi G}{3} \frac{a a_0 (\rho + 3P_T)}{r_0}} - \frac{a a_0}{2} \frac{\Lambda}{3}, \tag{1.69}$$

which should be taken as one of the Friedmann's equations in the MONDian case. There are two large differences with respect to the standard second order Friedmann equation. The first one is that the dark energy term is a function of the acceleration constant. This particularity will not affect qualitatively the expansion. The most worrying difference, is that it is *not* possible to factor out the constant r_0 as was made in the Newtonian derivation in section 1.1. Thus, the expansion represented by this equation is a function of the initial scale assumed for the system.

To solve the equation for a particular cosmological model, for instance an Einstein-de Sitter universe, will help in quantifying the importance of this new effect. Neglecting pressure terms and using the solution of the continuity equation (eq.1.9):

$$\rho = \frac{\Omega_0 \rho_c}{a^3} = \frac{3H_0^2}{8\pi G} \frac{1}{a^3} \tag{1.70}$$

one can rewrite eq.1.69 as:

$$\ddot{a} = \frac{1}{a} \sqrt{\frac{a_0 H_0^2 \Omega_0}{2r_0}}. \tag{1.71}$$

A first integral can be obtained analytically using, as in the Newtonian case, $\dot{a}$ as integrating factor. The solution is:

$$\dot{a} = 2\sqrt{\frac{a_0 H_0^2 \Omega_0}{2r_0}} \ln(a) + A, \tag{1.72}$$

where A is the integration constant. A second large difference with respect to Newtonian cosmology appears: the expansion is not only function of the scale but any density parameter gives a closed universe under MOND, with different time scale for re-collapse for each spatial scale.

The second integration that is required to get the solution can be made numerically. The result is shown in fig.1.1 for A=0 and Ω_0=1 for three different scales. The Newtonian solution is also shown for comparison. The problem is not only formally in the equation, but the numbers show that it is not a negligible effect. Changing Ω_0 into smaller values does not solve the problem (less mass can be read as more MONDian, which will not change the effect).

It seems that we arrive to a dead end at least for the Newtonian set of arguments that we wanted to build. An examination of the assumptions made during the analysis could help to understand the reasons for the reasoning to fail. The ingredients employed in the analysis are the following ones:

- The Birkoff theorem was assumed as valid. One should not forget that there are no solutions for Poisson equation for an infinite isotropic and homogeneous density

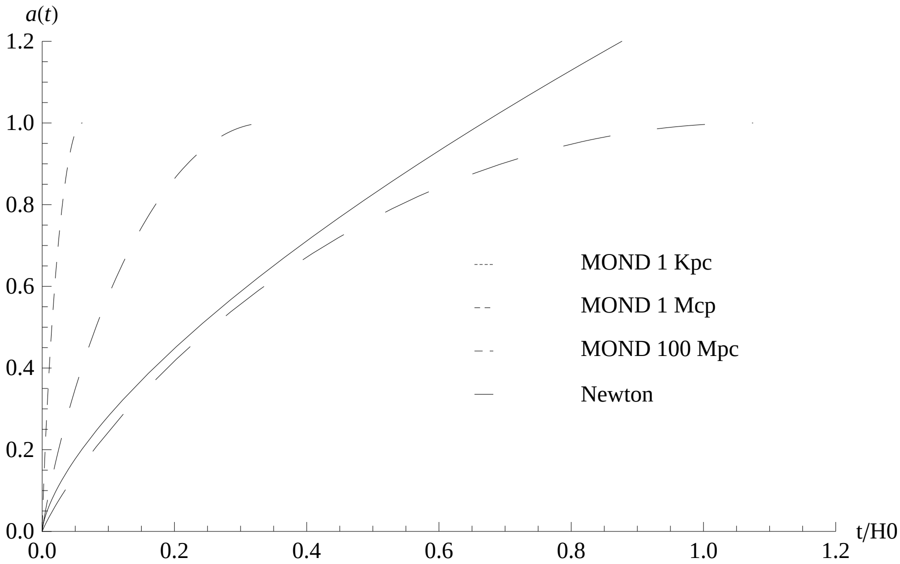


Figure 1.1: Friedman solutions for $\Omega_0 = 1$ for Newtonian (continuous curve) and MONDian (dashed curves) cases for three different spatial scales. Time is in units of $H_0 = 100$ km/sec/Mpc.

field and that Birkoff theorem is in fact a relativistic result that should be confirmed using a relativistic extension of MOND.

- The continuity equation was also accepted to be valid, but section 1.2 shows that this result is no more than a different way to write the standard Friedmann's equations. In other words, solutions from a different theory are beeing included in the analysis without further justification.

- The calculation was made in the deepMOND regime. To relax this assumption will not change the qualitative behavior of the solution.

- Extra fields that could be needed in relativistic extensions of MOND were neglected. To add this fields should not change the behavior of the theory on the scale of galaxies (the theory was designed to explain galaxies without the addition of any extra field), but expansion is a large scale phenomena, where the extra fields could have an impact.

When thinking about expansion, one also must take into account that the measurement of the expansion is made through the standard model. There are long term projects to measure it in a direct way (e.g. Liske et al. 2008; Pasquini et al. 2006), which will give a theory independent answer on this topic.

See Sanders (1998) for a detailed analysis on the solution of the equations discussed here and Felten (1984) for a discussion on the assumptions underlying this kind of approach.

Relativistic results

A better clue of what could be the behavior of the expansion in a MONDian universe can be found extending the study to other formulations for MOND. For instance, Sanders (2001) has proved using a bi-scalar classical Lagrangian can there are no MOND effects in the absence of density fluctuations. This result implies that the relation between the scale factor and the time is given by the standard Friedmann equations. Relativistic extensions as for instance Sanders (1997) or Halle et al. (2008) give same result, suggesting that this is a common feature and that only the fluctuations should be taken into account for MOND.

1.4.2 Linear evolution of density perturbations

Assuming a FRW background and using a spherical collapse model, Sanders (2001) derived an equation equivalent to eq.1.38 for the linear growth of perturbations under MOND. The equation is the following one:

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3\Omega_m}{2a^3}\delta = \frac{3g_2}{a\lambda_c}, \tag{1.73}$$

where g_2 is the MONDian contribution to the acceleration field and λ_c is a length scale.

Two large differences are present between this equation and the one studied in section 1.2.2. In first place, equation 1.73 is non-linear, so it is not possible for find analytic solutions. When applying a Runge-Kutta algorithm to obtain the solution, Sanders found that it is not possible to write down a separable solution as in the Newtonian case. In other words, different length scales evolve with different speeds. The rate of evolution is also different, being almost exponential at the moment in which the systems enter in the MONDian regime. In late times, due to the presence of a large cosmological constant, the speed of the collapse decreases. The normalization σ_8 extracted from the final power spectrum of perturbations obtained with this method, exceeds by 50% the value for the ΛCDM model.

Nusser (2002) studied also the linear evolution of perturbation under MOND using similar arguments. In this case, the MONDian forces were calculated in the deepMOND regime and by means of the simple MOND formula (eq.1.60). Employing the hydrodynamic equations plus the Poisson equation in comoving coordinates (see chapter 2 for and explanation of this coordinates), Nusser wrote a growth equation for the accelerations instead of the overdensity:

$$\frac{\partial^2 \mathbf{g}_N}{\partial t^2} + 2H\frac{\partial \mathbf{g}_N}{\partial t} = \left(\frac{3}{2a}\Omega H^2 a_0\right)^{1/2} \frac{\mathbf{g}_N}{|\mathbf{g}_N|^{1/2}}. \tag{1.74}$$

The solution of this equation behaves in a decreasing monotonic way, with an asymptotic value

$$g = \frac{3}{10}a_0 < a_0. \tag{1.75}$$

According to this analysis, once the accelerations enter in the MONDian regime, they stay there until redshift $z = 0$. The solution for the density field associated with this accelerations is given by:

$$\delta \sim t^{4/3} \sim a^2. \tag{1.76}$$

Sanders (2008) has shown that there is an attractor in the space of solutions of eq.1.73 which gives a logarithmic slope of -1 to the power spectrum. Furthermore, the normalization of the spectrum at late times is independent of the employed initial conditions.

1.4.3 Non-linear evolution of density perturbations

Very few advances were made in the context of non-linear evolution and galaxy formation under MOND. Spherical collapse models were presented in Sanders (2008). By integrating the equations of motion for spherical shells, he found that it is possible to obtain virialized objects that resembles elliptical galaxies and that they follow well defined radius-velocity dispersion and mass-velocity dispersion relationships.

In the context of self-consistent non-spherical calculations, at present (excluding the work that is provided in this thesis) there are only two works in the literature (Nusser 2002; Knebe & Gibson 2004). Both of them were made assuming that MOND does not affect the expansion of the universe and thus, employing the standard Friedmann equations to describe the background. Regarding the field equation used to calculate the MONDian potential, both authors approximated the conservative MOND equation (eq.1.64) by the simple non-conservative MOND formula (eq.1.60). Both calculations took into account only collision-less matter. The main result (which agrees also with Sanders (2001) linear results) is that the collapse in MOND is too strong at late times and that, in the case of using a standard power spectrum at the initial redshift, the final power spectrum over-estimates the observed values. Nusser studied the consequences of varying a_0 over the normalization of the final power spectrum. On the other side, Knebe decided to change the normalization of the initial power spectrum to match the observed spectrum and extended the study to the properties of the collapsed objects. One of the highlights of that work is that MOND is unable to produce objects with masses similar to the mass of the Milky Way. This possible issue will be revisited in the chapter 2 of this thesis under the same conditions in which Knebe worked and in chapter 5 with a different set of assumptions. While this simulations are one step above spherical collapse simulations, there are still strong assumptions in the calculations that make obscure the understanding of the underlying physics. The main weak points are the absence of curl field effects, the treatment made of the initial conditions and the absence of baryonic physics and cooling.

1.5 This thesis

The standard cosmological model is very successful in explaining many aspects of cosmological evolution. Some examples of the predictions that were made by the model and that are consistent with observations are the abundance of elements given by the primordial nucleosintesis, the spectrum of the CMB radiation and the matter power spectrum at redshift $z = 0$. On the other side, there are still no strong predictions from the MONDian cosmological model, especially at low redshift. Thus, the main question we want to address in this thesis is the following: *can MOND reproduce the success of ΛCDM in a qualitative and quantitative way in the context of post-recombination cosmological evolution?*

Naturally, the standard model is not perfect. Besides the fact that the nature of the two unseen components is still not understood, there is a mismatch between predictions and observations in many different situations (e.g. cuspy cores, number of satellites, abundance in voids, etc). In case that these problems are the signal of a crisis and that we are indeed living a moment in which a Kuhnian paradigm shift is happening, the new paradigm must be able to solve these difficulties. In this context, this thesis is also intended to test how MOND performs when it is applied to the weak points of ΛCDM. In particular, three of the sensitive points of the standard model will be tackled: the void phenomenon, the colissional velocity of objects and the separation between centers in baryonic matter and lensing signal.

Other very important point that needs to be address is how to discern by means of observations between this two families of gravitational theories, i.e. those with Poissonian and MONDian weak limits. To this end, it is necesary to find the observables for which both theories give different predictions. This leads to one more of the goals of this thesis, which consist in the search for this kind of predictions in the context of non-linear structure formation.

As was discussed in section 1.4, there are very few works intended to study non-linear evolution with MOND (Nusser 2002; Knebe & Gibson 2004). Both of them studied toy models in which many approximations were made, therefore, to give strength to the predictions obtained using this kind of simulations, it is still necessary to refine the techniques and to test the validity of the adopted hypothesis. Consequently, a large part of the work will be dedicated to discussion of the technical issues involved and to the improvement of the methods.

One of the main assumptions that will be relaxed with respect to previous works in structure formation with MOND is the absence of curl field effects. While it is true that the curl term in the Bekenstein-Milgrom (BM) equation (eq.1.65) decreases very fast with the distance for isolated objects, it still could affect the dynamics in the highly non symmetric situations which are present for instance in the environment of satellites galaxies or in the cosmic web it self. The techniques that will be employed here to solve the BM equation were already used with satisfactory results in the context of galactic dynamics (Brada & Milgrom 1999; Tiret & Combes 2007), but never implemented in a cosmological context. As the technical complexity strongly increases when dealing with the BM equation, the study of the influence of pressure and cooling effects will be left for future work. Thus, it is important to emphasize that during this thesis, the analysis will be focused only on the evolution of dust, i.e. collision-less matter. Nusser (2002) already found, employing dust simulations, that strong anti-bias could be needed to match the normalization of the power spectrum at redshift $z = 0$. The simplification that will be made in this sence should be taken into account when extracting conclusions from the results presented here.

More care will be taken also when generating the initial conditions for the simulations. It is commonly accepted that the universe at high redshift can be understood using Newtonian physics. This assumption permits to use standard algorithms to determine the initial position of the particles. The validity of this hypothesis will be tested (with negative results) and an improved method to generate initial conditions for cosmological simulations will be proposed.

The equations needed to describe the evolution of dust and the methods that will be used to solve them will be developed under the same line of ideas presented in section 1.2.3. However, there will be a very important difference in our treatment. The fact that there is no unique relativistic extension of MOND makes any kind of relativistic analysis theory dependent and entails the risk of loosing generality and missing the fundamental points that characterize the MOND theory. Instead of deriving the equations from basic principles as was done for the standard case in section 1.2.1, a minimalistic and more general approach will be taken. For the evolution of the background, the same approach used by other authors (e.g. Sanders 2001; Nusser 2002; Knebe & Gibson 2004, etc.) will be adopted: it will be assumed that MOND does not affect the expansion and that the standard Friedmann's equations are valid. Regarding the field equation for the metric perturbations, the derivation will be made using a set of standard Newtonian arguments, with subtle differences that will be motivated by the non-linearity of the BM equation. Furthermore, any extra field necessary to extend the MOND formula into the relativistic framework will be neglected in large part of this work. Only the last chapter will be dedicated to the study of the influence of extra fields in a particular case of bimetric theories.

In chapter 2, we will study the effects of the curl field that was neglected in the previous literature. We will start the chapter developing a non-relativistic formalism from which the equations for the accelerations in an expanding background can be derived. To this end, the minimalistic approach described above will be adopted. The chapter will continue with a description of the method that will be used to solve the BM equation and of the tests that were made to the implementation. Finally, we will repeat the calculations presented in Knebe & Gibson (2004) using the new set of equations and codes. The results will be compared to understand the characteristics of the curl field it self and its consequences on the non-linear cosmological evolution.

Chapter 2 will also include a strong test that was made to the code (and to the theory) that consists in the verification of stability of triaxial models for galaxies. The models where previously developed by Wang et al. (2008) using the Schwarschild method.

In chapter 3 we will study the velocity field of the simulation presented in chapter 2. The original motivation of the chapter was to repeat the analysis presented in Hayashi & White (2006), but in the context of MOND. That work deals with the probability of having a collision of clusters as fast as the one present in the Bullet Cluster (Clowe et al. 2006). As the box size used in the simulation employed here is too small to include clusters, an extrapolation is needed. We propose a normalization of the velocities that gives results independent of the mass of the objects and find strong differences between MONDian and Newtonian simulations, which could give the chance to distinguish the two theories using observations.

Chapter 4 will be dedicated to the study of the concept of phantom dark matter. This idea was introduced very early in the literature (Milgrom 1986) and refers to the amount of dark matter needed to reproduce a MONDian potential in a Newtonian context. We describe the way in which this quantity can be calculated on a cosmological context and show some properties of its distribution in the simulations presented in chapter 2.

Chapter 5 will revisit the generation of initial conditions for non-linear calculations. The standard argument to generate initial conditions for MOND is that the accelerations at high redshift are high enough to be considered as Newtonian (e.g. Nusser 2002; Knebe & Gibson 2004; Angus 2009) and hence standard algorithms with standard gravity can be employed. We will test this hypothesis by measuring the accelerations in a direct way on realizations of density fields generated at different redshifts. As the results of the test will be negative (accelerations are in fact MONDian at very high redshift), we will propose a method to generate initial conditions using MONDian gravity. Part of the

ID	Ω_b	Ω_c	Ω_Λ	Ω_ν	H_0	$\Omega_{\rm tot}$	Comments
A	.053	.206	.741	0	72.4	.259	ΛCDM
B	.053	0	.947	0	72.4	.053	ΛCDM without dark matter
C	.04	.04	0	0	70.0	.08	KG04
D	.04	.04	.92	0	70.0	.08	KG04 with dark energy
E	.047	0	.724	.229	71.5	0.276	Angus (2009) with neutrinos

Table 1.1: Cosmological models shown in Fig.1.3 and Fig.1.2. See text for explanation of the motivation for each model. The neutrinos used in the model E have a mass of 11 eV.

calculations will be made in this case using a new conservative formulation for MOND proposed in Milgrom (2010b). The great advantage of this formulation is that the field equation is given by a system of two linear Poisson equation with different source. This opens the possibility to get solutions using standard gravity solvers based on Fourier methods.

In chapter 6 we will study the concept of twin matter proposed recently in Milgrom (2010a). This is a reinterpretation of MOND in the context of a bimetric relativistic theory, which describe MOND as the interaction between two fluids with repulsive behavior. We will describe the implementation of the equations in a particle mesh code that was developed from scratch during this thesis. The code was made with the intention to have a place where to make preliminary test of new theories. Consequently, it was designed to be small and simple, which makes possible to introduce the necessary modifications for different gravitational models in a quick and safe way, minimizing the probability of introducing bug in the process. The results obtained with this code, can be used as reference for modifications made a posteriori in bigger codes as for instance Ramses (Teyssier 2002) or Gadget (Springel 2005). We will show the behavior of the time evolution of the power spectrum for this new theory and a few characteristics of the interaction between the two fluids.

Finally in chapter 7, we summarize the basic results and address the questions presented here.

The present chapter will be closed with a discussion about the cosmological models that are associated to MOND (i.e.which is the way in which energy is distributed between its different components). The original proposal by Milgrom was to assume that the universe is filled only with baryons, radiation and a possible cosmological constant. The theory was designed to explain galactic rotation curves under this assumption. Thus, on those scales, there should not be perturbations in any dark sector. There is a possible drawback associated with this model: under the assumption that the evolution of the early universe is dictated by Einstein's gravity, the model is unable to fit the angular power spectrum of perturbations on the CMB (McGaugh 2004). There are two possible ways to go around the problem. On one side, one can assume than non standard relativistic effects as, for instance, the coupling between matter and a vector field, will have an impact on the evolution of the universe even in the case that the accelerations are Newtonian, as it is assumed at early epochs. In other words, the mismatch between the data and the cosmology proposed by Milgrom could be a consequence of the fact that

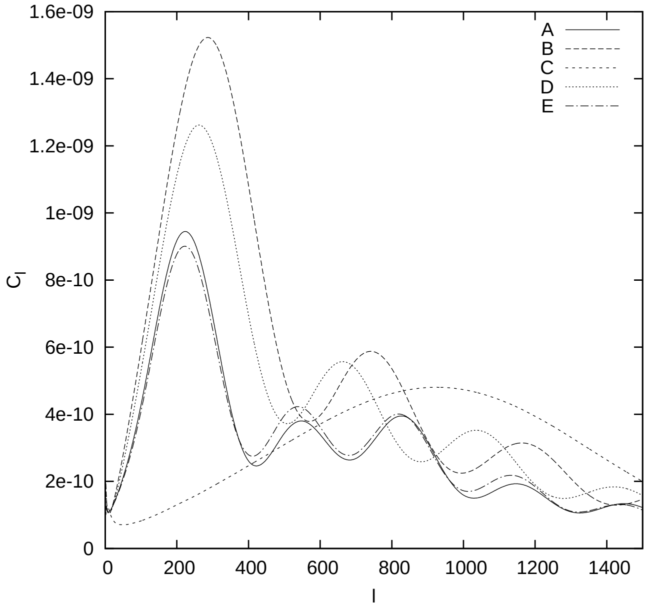


Figure 1.2: Angular power spectrum of perturbations for the models described in table 1.1.

the theory is still incomplete.

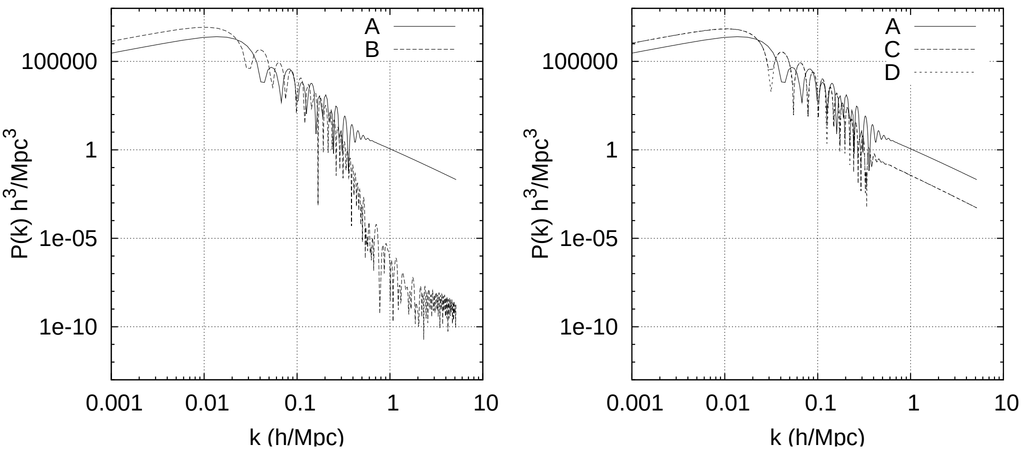
Another way to improve the fitting of the CMB data is to assume that Einstein's equation is valid in the early universe and to include a neutrino component in the energy distribution. As was discussed in section 1.1, classical neutrinos are inconsistent with cluster's data. Nevertheless, a promising model have appeared during the calculations for this thesis (Angus 2009) using 11eV massive neutrinos as the dark matter component. This proposal can solve the inconsistencies found on the CMB data and on clusters of galaxies and at the same time lives the high frequencies region of the power spectrum unperturbed to let MOND to make the baryons to collapse on those scales building galaxies.

As an example of the effect of different non-standard cosmologies on the CMB angular power spectrum, Fig.1.2 shows the angular power spectrum calculated with standard gravity with the code cmbfast (Seljak & Zaldarriaga 1996)[4] for the cosmologies shown in table 1.1. Fig.1.3 shows the power spectrum of baryon perturbations calculated with the same code at the redshift of the CMB (z=1000). The motivation for each model included in the table and some characteristics of their associated spectra are the following ones:

- Model A: standard ΛCDM parameters proposed in Angus (2009). It is shown here as a good approximation to the observations.

- Model B: it is the most straightforward flat cosmology that one can thing of in a universe populated only with baryons as Milgrom proposed originally. It does not fit the angular power spectrum. The acoustic oscillations are in different place in the power spectrum.

[4]See Callin (2006) for an interesting discussion about CMB calculations.



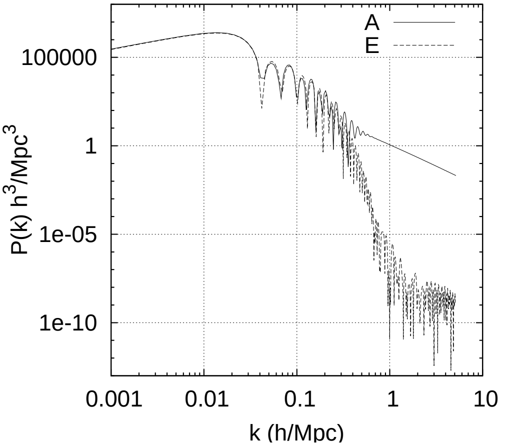


Figure 1.3: Power spectrum of baryonic perturbations at redshift 1000 for the models show in table 1.1. Up-left: simple model with only a baryonic component. Up-right: Knebe & Gibson (2004) models (open and closed models lay in top of each other). Down: Angus (2009) model. In all the panels, the standard ΛCDM power spectrum is shown as a comparison. The normalization is arbitrary.

- Model C: cosmology proposed in Knebe & Gibson (2004) (KG04) that will be discussed in chapter 2. It includes a low density dark matter component that makes small scales to collapse and gives a power law form with slope -3 to the power spectrum. This new component decrease the amplitude of the wiggles, but they are still in different place than in ΛCDM. The angular power spectrum is still far for the ΛCDM spectrum.

- Model D: a flat version of the model C. To include a cosmological constant does not improve the fitting of the angular power spectrum.

- Model E: Model proposed in Angus (2009) with a massive neutrino component. It conserves Milgrom's original proposal (no dark matter on the scales of galaxies) while greatly improves the fitting of the angular power spectrum. Owing to the dark matter equivalent behavior of the neutrinos on large scales, the wiggles are in the same place that in ΛCDM. This cosmology has also the property to give good fitting to clusters data (Angus et al. 2009).